Fusion v mutation

$$\begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 3 \\ \hline\end{array} + 2\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|}\hline 2 \\ \hline\end{array}^{2} + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array}^{2} + 2\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline\end{array} \cdot \begin{array}{|c|}\hline 3 \\ \hline\end{array}$$

Have you seen this product?

We present a cluster structure on semi-standard Young tableaux.
Based on arXiv:2106.07101 with Roger Bai and Joel Kamnitzer.

Plan

- $\mathrm{Gr}^\lambda \times \mathrm{Gr}^\lambda \to \mathrm{Gr}^\lambda \tilde{\times} \mathrm{Gr}^\lambda \to \mathrm{Gr}$
- item 2
- item 3
  - here we are
  - and there you go
- subitems in the last item

Example

Let $\tau' = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}$, $\tau'' = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline\end{array}$. The requirement that a matrix of the form

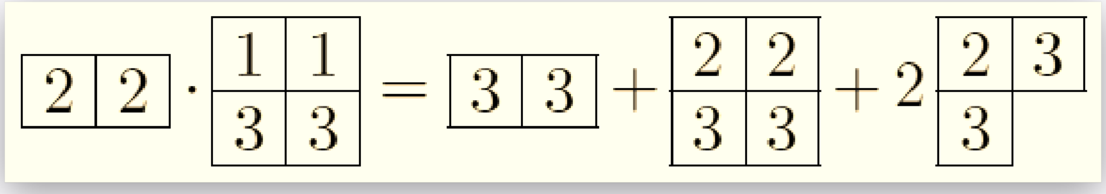
$$\begin{bmatrix} 0 & 1 & & & & \\ -s^2 & 2s & A^1_{12} & A^2_{12} & A^1_{13} & A^2_{13} \\ & & 0 & 1 & & \\ & & & 0 & A^1_{23} & A^2_{23} \\ & & & & 0 & 1 \\ & & & & -s^2 & 2s \end{bmatrix}$$

is contained in the geometric fusion $X(\tau') *_{\mathbb{A}^\times} X(\tau'')$...

...forces the equations

$$
\begin{aligned}
&2A^2_{12}A^2_{23}s + 3A^2_{13}s^2 + A^2_{12}A^1_{23} + A^1_{12}A^2_{23} + 2A^1_{13}s,\; A^2_{13}s^3 - A^2_{12}A^1_{23}s - A^1_{12}A^2_{23}s - 2A^1_{12}A^1_{23},\\
&A^2_{12}A^2_{13}A^1_{23}s^2 + A^1_{12}A^2_{13}A^2_{23}s^2 - (A^2_{12}A^1_{23})^2 + 2A^1_{12}A^2_{12}A^1_{23}A^2_{23} - (A^1_{12}A^2_{23})^2 + 6A^1_{12}A^2_{13}A^1_{23}s + 4A^1_{12}A^1_{13}A^1_{23},\\
&A^1_{12}A^2_{13}(A^2_{23})^2s^2 - (A^2_{12}A^1_{23})^2A^2_{23} + 2A^1_{12}A^2_{12}A^1_{23}(A^2_{23})^2 - (A^1_{12})^2(A^2_{23})^3 - 2A^2_{12}A^2_{13}(A^1_{23})^2s\\
&\quad + 4A^1_{12}A^2_{13}A^1_{23}A^2_{23}s - A^1_{13}A^2_{13}A^1_{23}s^2 - 3A^1_{12}A^2_{13}(A^1_{23})^2 + 4A^1_{12}A^1_{13}A^1_{23}A^2_{23},\\
&(A^2_{12})^3(A^1_{23})^2A^2_{23} - 2A^1_{12}A^1_{23}(A^2_{12}A^2_{23})^2 + A^2_{12}(A^1_{12})^2(A^2_{23})^3 + 2A^2_{13}(A^2_{12}A^1_{23})^2s + 2A^2_{13}(A^1_{12}A^2_{23})^2s\\
&\quad + 3A^1_{12}(A^2_{13})^2A^1_{23}s^2 + A^1_{13}(A^2_{12}A^1_{23})^2 + 4A^1_{12}A^2_{12}A^2_{13}(A^1_{23})^2 - 6A^1_{12}A^2_{12}A^1_{13}A^1_{23}A^2_{23}\\
&\quad + 4(A^1_{12})^2A^2_{13}A^1_{23}A^2_{23} - 4A^1_{12}A^1_{13}A^2_{13}A^1_{23}s - 4A^1_{12}(A^1_{13})^2A^1_{23} + A^1_{13}(A^1_{12}A^2_{23})^2
\end{aligned}
$$

to vanish. Setting $s = 0$ results in a union of three irreducible components which we record as the combinatorial fusion



The slice and the orbit

$O^\lambda \subset M_N(\mathbb{C})$ denotes the nilpotent (adjoint) orbit of matrices A having Jordan type λ or $rk A^c = N - \#$boxes in first c columns of λ for all c.

T_μ denotes the affine space of $\mu \times \mu$ block matrices $J_{0,\mu} + X$ where X is a block matrix with possibly nonzero entries in the first $\min(\mu_i, \mu_j)$ columns of the last row of each $\mu_i \times \mu_j$ block.

E.g. If $\mu = (3, 2)$ then T_μ looks like

$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ A_{11}^1 & A_{11}^2 & A_{11}^3 & A_{12}^1 & A_{12}^2 \\ \hline 0 & 0 & 0 & 0 & 1 \\ A_{21}^1 & A_{21}^2 & 0 & A_{22}^1 & A_{22}^2 \end{array}\right]$$

Generalized orbital varieties

Given $A \in M_N(\mathbb{C}) \cap n$ we denote by $A\Big|_{\mathbb{C}^p}$ the restriction of A to the subspace spanned by the first p standard basis vectors of $\mathbb{C}^N$ and identify it with the $p \times p$ upper-left submatrix of A.

Let $\mu \leq \lambda \vdash N$. Given $\tau \in YT(\lambda)_\mu$ have $\tau\Big|_{\{1,\ldots,i\}} \in YT(\lambda(i))_{\mu(i)}$ and

$$\mathring{X}(\tau) := \{A \in T_\mu \cap n : A\Big|_{\mathbb{C}^{|\mu(i)|}} \in O^{\lambda(i)} \text{ for each } i = 1, \ldots, m\}$$

Denote by $X(\tau)$ the top-dim irreducible component of its closure.

Theorem. $\{X(\tau)\}_{\tau \in YT(\lambda)_\mu}$ are irreducible components of $\overline{O}^\lambda \cap T_\mu \cap n$.

Two-point deformation of $T_\mu \cap n$

$U_{0,\mathbb{A}}^{\mu',\mu''}$ is the family of pairs $(A, s) \in T_\mu \times \mathbb{A}$ such that A is weakly block upper-triangular with diagonal equal to the companion matrices of the polynomials $t^{\mu_i'}(t-s)^{\mu_i''}$

E.g. $U_{0,s}^{(1,1,0),(2,1,1)}$ looks like

$$\left[\begin{array}{ccc:cc:c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -s^2 & 2s & A_{12}^1 & A_{12}^2 & A_{13}^1 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & s & A_{23}^1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & s \end{array}\right]$$

Two-point deformation of $\overline{O}^{\lambda}$

For $s \neq 0$ the (adjoint) orbit of matrices conjugate to $J_{0,\lambda'} \oplus J_{s,\lambda''}$ or

$$O_{0,s}^{\lambda',\lambda''} := \left\{ A \in M_N(\mathbb{C}) : \begin{array}{r} rkA^c = N - \#\text{boxes in first } c \text{ columns of } \lambda', \\ rk(A-s)^c = N - \#\text{boxes in first } c \text{ columns of } \lambda'', \\ \text{for all } c \geq 0 \end{array} \right\}$$

Fact. There exists a flat family $\overline{O}_{0,\mathbb{A}}^{\lambda',\lambda''} \to \mathbb{A}$ whose fibre over $s \in \mathbb{A}$ is reduced and given by $\overline{O}_{0,s}^{\lambda',\lambda''}$ if $s \neq 0$ and $\overline{O}^{\lambda}$ if $s = 0$.

Two-point deformation of $X(\tau)$

For $s \neq 0$ and a pair of tableaux $\tau' \in YT(\lambda')_{\mu'}$, $\tau'' \in YT(\lambda'')_{\mu''}$ define

$$\mathring{X}(\tau', \tau'')_{0,s} = \left\{ A \in U_{0,s}^{\mu', \mu''} : A\big|_{\mathbb{C}^{|\mu(i)|}} \in O_{0,s}^{\lambda'(i), \lambda''(i)} \text{ for } i = 1, \dots, m \right\}$$

and

$$\mathring{X}(\tau', \tau'')_{0,\mathbb{A}^\times} = \left\{ (A, s) \in M_N \times \mathbb{A}^\times : A \in \mathring{X}(\tau', \tau'')_{0,s} \right\}$$

and $X(\tau', \tau'')_{0,\mathbb{A}}$ to be the closure of its top-dim component in $M_N \times \mathbb{A}$.

Geometric fusion

Define the fusion of generalized orbital varieties as the scheme-theoretic intersection

$$X(\tau', \tau'')_{0,\mathbb{A}} \cap M_N \times \{0\} =: X(\tau', \tau'')_{0,0}$$

in $M_N \times \mathbb{A}$.

Fact. $X(\tau', \tau'')_{0,0}$ is contained in $\overline{O}^{\lambda} \cap T_\mu \cap n$.

The intersection multiplicities $i(X(\tau), X(\tau', \tau'')_{0,0})$ are the structure constants of our algebra of tableaux.

What is this mystery algebra?

In type A the geometric Satake correspondence $(*)$ is about tableaux.

$$\begin{array}{ccccc}
\bigcup YT(\lambda) & \xrightarrow{\tau \mapsto X(\tau)} & \bigcup \mathcal{X}(\lambda) & \xrightarrow[*]{X \mapsto v_X} & \bigcup V(\lambda) \\
\Big\downarrow \mathsf{L} & & \Big\downarrow & & \Big\downarrow {\scriptstyle v \mapsto b_v} \; \Psi \\
\mathbb{N}^r & = & B(\infty) & \longleftarrow & \mathbb{C}[N]
\end{array}$$

Fact. $b(\tau) \in \mathbb{C}[N]$ coming from the generalized orbital varieties form a *perfect basis* with structure constants $i(X(\tau), X(\tau', \tau'')_{0,0})$. We call this the MV basis.

Remarks

- commutativity of geometric fusion
- combinatorial fusion often symmetrized RSK

Cluster algebras

$\mathbb{C}[N]$ is a cluster algebra: by mutating from an initial seed $(\{x_1, \ldots, x_r\}, B)$ of $r = m(m-1)/2$ cluster variables $x_i \in \mathbb{C}[N]$, meaning mutable x_i are replaced by x_i^* according to the exchange relation

$$x_i x_i^* = x_+ + x_-$$

with $x_\pm$ denoting monomials in $\{x_j : j \neq i\}$ determined by B, we obtain the cluster variables.

Each mutation gives a new cluster. The *cluster monomials* are products of cluster variables that are supported on a single cluster.

Motivating conjecture

$\mathbb{C}[N]$ has three interesting (biperfect) bases: the MV basis, the dual canonical basis, and the dual semicanonical basis. They disagree at a point which is not a cluster monomial.

Conjecture. The cluster monomials are contained in the MV basis.

Given an initial seed of "MV cycles" if we can find (for each i) an MV cycle $X(\tau_i^*)$ such that $X(\tau_i) * X(\tau_i^*) = X(\tau_+) \cup X(\tau_-)$ then we can conclude that $b(\tau_i^*) = x_i^*$ and (since all cluster variables are created via exchange) deduce that the conjecture is true.

Thank you for listening

Example 2.6.5. Let D be an effective Cartier divisor on a scheme X, i the inclusion of D in X. Let Y be a closed subscheme of X. Assume that Y is purely n-dimensional and $D \cap Y$ has dimension $n-1$. Let $V_1, \ldots, V_r$ be the (reduced) irreducible components of $D \cap Y$. Let A_i be the local ring of Y along V_i, and let a_i be a local equation for D in A_i. Then

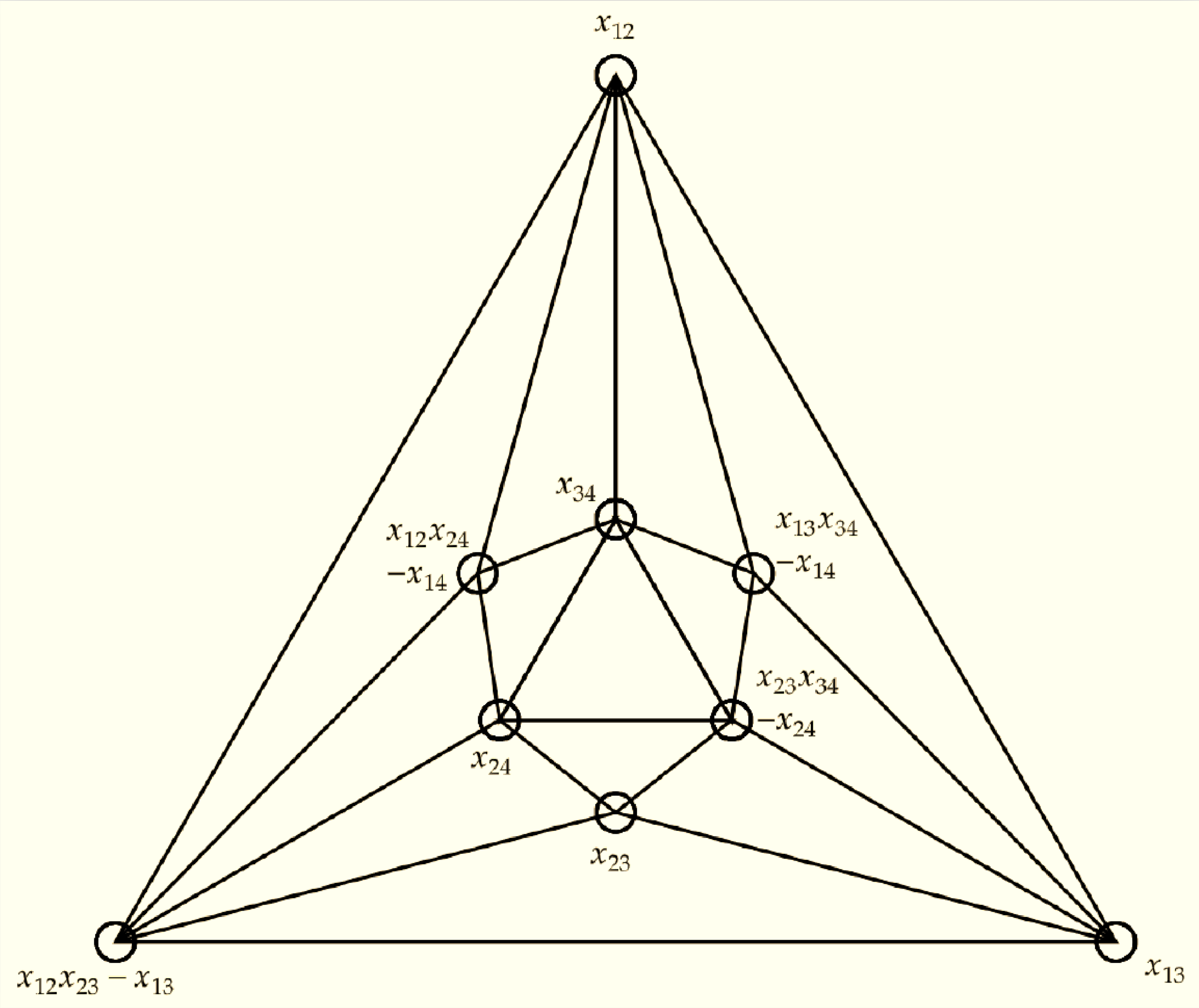
$$i^*[Y] = D \cdot [Y] = \sum_{i=1}^{r} e_{A_i}(a_i, A_i)\,[V_i]$$

where $e_{A_i}(a_i, A_i)$ is the multiplicity defined in Appendix A.2. More generally, if one assumes only that $\dim Y \leqq n$ and $\dim D \cap Y \leqq n-1$, and $V_1, \ldots, V_r$ are the components of $D \cap Y$ of dimension $n-1$, then the right side of this equation gives a formula for $i^*([Y]_n)$, where $[Y]_n$ is the n-dimensional component of the cycle of Y. (These formulas follow from Lemma A.2.7.)

12 cluster variables

$(1,0,0,0,0,0)$	$\boxed{2}$	$\mathbb{C}[A^1_{12}]$	$\mathbb{P}^1$	x_1
$(0,0,0,1,0,0)$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{23}]}{(A^1_{12},A^1_{13})}$	$\mathbb{P}^1$	$\frac{x_2+x_3}{x_1}$
$(0,0,0,0,0,1)$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^2_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^2_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24})}$	$\mathbb{P}^1$	$\frac{x_2x_4+x_3x_6+x_1x_5}{x_2x_3}$
$(1,0,0,1,0,0)$	$\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{23}]}{(A^1_{23})}$	$\mathbb{P}^2$	x_2
$(0,1,0,0,0,0)$	$\boxed{3}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{23}]}{(A^1_{12})}$	$\mathbb{P}^2$	x_3
$(0,0,0,1,0,1)$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{14},A^1_{34})}$	$\mathbb{P}^2$	$\frac{x_1x_5+x_6(x_2+x_3)}{x_1x_2}$
$(0,0,0,0,1,0)$	$\begin{array}{\|c\|}\hline 1\\\hline 4\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{23},A^1_{34})}$	$\mathbb{P}^2$	$\frac{x_1x_5+x_4(x_2+x_3)}{x_1x_3}$
$(0,1,0,0,0,1)$	$\begin{array}{\|c\|c\|}\hline 1 & 3\\\hline 2 & \\\cline{1-1} 4 & \\\cline{1-1}\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{34},A^1_{13}A^1_{24}-A^1_{23}A^1_{14})}$	$S(2,4)$	$\frac{x_3x_6+x_1x_5}{x_2}$
$(1,0,0,0,1,0)$	$\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^2_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{23},A^1_{24})}$	$S(2,4)$	$\frac{x_2x_4+x_1x_5}{x_3}$
$(0,0,1,0,0,0)$	$\boxed{4}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{13},A^1_{23})}$	$\mathbb{P}^3$	x_4
$(0,1,0,0,1,0)$	$\begin{array}{\|c\|}\hline 3\\\hline 4\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{12},A^1_{34})}$	$Gr(2,4)$	x_5
$(1,0,0,1,0,1)$	$\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$	$\frac{\mathbb{C}[A^1_{12},A^1_{13},A^1_{14},A^1_{23},A^1_{24},A^1_{34}]}{(A^1_{23},A^1_{24},A^1_{34})}$	$\mathbb{P}^3$	x_6

3 mutable variables per cluster for a total of 9, plus 3 frozen



For an $R(\beta)$-module M and an $R(\gamma)$-module N, we define the *convolution product* $M \circ N$ by

$$M \circ N = R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N).$$

Definition 2.2.5. *For simple R-modules M and N, we denote by $M \nabla N$ the head of $M \circ N$ and by $M \Delta N$ the socle of $M \circ N$.*

Definition 3.1.1. *For non-zero $M, N \in R\text{-gmod}$, we denote by $\Lambda(M,N)$ the homogeneous degree of the R-matrix $\mathbf{r}_{M,N}$.*

Corollary 4.1.2. *Let M and N be simple modules. We assume that one of them is real. Assume that M and N do not commute, Then we have the equality in the Grothendieck group $K(R\text{-gmod})$*

$$[M \circ N] = [M \nabla N] + [M \Delta N] + \sum_k [S_k]$$

with simple modules S_k. Moreover we have the following:

(i) *If M is real, then we have $\Lambda(M, M \Delta N) < \Lambda(M, N)$, $\Lambda(M \nabla N, M) < \Lambda(N, M)$ and $\Lambda(M, S_k) < \Lambda(M, N)$, $\Lambda(S_k, M) < \Lambda(N, M)$.*

(ii) *If N is real, then we have $\Lambda(N, M \nabla N) < \Lambda(N, M)$, $\Lambda(M \Delta N, N) < \Lambda(M, N)$ and $\Lambda(N, S_k) < \Lambda(N, M)$, $\Lambda(S_k, N) < \Lambda(M, N)$.*

Rank Varieties of Matrices

David Eisenbud and David Saltman

Abstract. In this paper we extend work of Gerstenhaber [5], Kostant [9], Kraft-Procesi [10], Tanisaki [15], and others on orbit closures in the nilpotent cone of matrices by studying varieties of square matrices defined by conditions on the ranks of powers of the matrices, or more generally on the ranks of polynomial functions of them. We show that the irreducible components of such varieties are always Gorenstein with rational singularities (in particular they are normal). We compute their tangent spaces, and also their limits under deformations of the defining polynomial functions. We also study generators for the ideals of such varieties, and we compute the singular loci of the hypersurfaces in the space of $n \times n$ matrices given by the vanishing of a single coefficient of the characteristic polynomial.

In the last section we saw that if r is a rank function, then

$$X_r := \{A \in \mathrm{End}(V) \,|\, \operatorname{rank} A^k \leq r(k)\}_{\mathrm{red}}$$

is a normal variety. In this section we will prove that it is in fact Gorenstein with rational singularities. We will also show that it fits into a flat family over $\mathbb{A}^m$ of normal varieties, whose fiber over a point $(\lambda_1, \ldots, \lambda_m)$ such that the λ_i are all distinct is

$$X_{r;\lambda_1,\ldots,\lambda_m} = \{A \in \mathrm{End}(V) \,|\, \operatorname{corank}(A - \lambda_i) \geq r(i-1) - r(i), i = 1, \ldots, m\}.$$

THEOREM 2.1.

i) $\mathcal{X}_{r,W}$ *is smooth over* W *and irreducible, of dimension* $n^2 - \sum_i a_i(r)^2 + \dim W$, *with trivial canonical bundle, while* π_1 *is proper and birational, with*

$$R^i \pi_{1*} \mathcal{O}_{\mathcal{X}_{r,W}} = 0 \text{ for } i > 0.$$

ii) $X_{r,W}$ *is normal, so that* $X_{r,W} = X'_{r,W}$, $\mathcal{X}_{r,W}$ *is a rational resolution of singularities of* $X_{r,W}$, *and* $X_{r,W}$ *is Gorenstein.*

iii) $X_{r,W}$ *is the restriction of* $X_{r,\mathbf{A}^m}$ *to* W, *and it is flat over* W.

We shall reformulate Theorem 4.1 in terms of generalized Gelfand-Tsetlin patterns which will be defined separately for each type. First we recall familiar Gelfand-Tsetlin patterns.

A *GT-pattern* (or $g\ell_r$-pattern) is an array $\Lambda = (\lambda_{ij})$ $(1 \leq i \leq j \leq r)$ of nonnegative integers λ_{ij} satisfying $\lambda_{i,j-1} \geq \lambda_{i+1,j} \geq \lambda_{ij}$ for all $1 \leq i < j \leq r$. It is usually drawn as follows:

$$\Lambda = \begin{bmatrix} \lambda_{11} & & \lambda_{12} & \cdots\cdots\cdots\cdots & & \lambda_{1r} \\ & \lambda_{22} & & \lambda_{23} \cdots\cdots\cdots & \lambda_{2r} & \\ & & \cdot & & \cdot & \\ & & & \cdot \quad \cdot & & \\ & & & \cdot \;\; \cdot & & \\ & & & \lambda_{rr} & & \end{bmatrix}$$

The vector $\lambda = (\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1r}) \in \mathbb{Z}^r$ will be called the highest weight of Λ, and the weight $\beta = (\beta_1, \ldots, \beta_r)$ of Λ is defined by $\beta_i = |\lambda_i| - |\lambda_{i+1}|$, where $|\lambda_i| = \lambda_{ii} + \lambda_{i,i+1} + \ldots + \lambda_{ir}$, $|\lambda_{r+1}| = 0$.